Chapter 9 **(Oechslin: 8.12)**

Hahn's orrery in Furtwangen (1774)

This chapter is a work in progress and is not yet finalized. See the details in the introduction. It can be read independently from the other chapters, but for the notations, the general introduction should be read first. Newer versions will be put online from time to time.

9.1 Introduction

The orrery described here is located in Furtwangen and was constructed in 1774 by Philipp Matthäus Hahn (1739-1790).[1]

It was acquired by Charles Frederick (1728-1811), margrave of Baden, and it was used for teaching in the *Gymnasium illustre*, a school initially founded in Durlach, and that moved to Karlsruhe.[2] This is now part of the University of Karlsruhe. The orrery came to Furtwangen in 1865.

It is the only "pure" orrery by Hahn, all his other clocks and machines contain other parts, such as globes. Nevertheless, there are three other clocks or machines by Hahn that contain similar orreries. The first is that of the Ludwigsburg *Weltmaschine* (Oechslin 8.1) from 1769, now in Stuttgart. After the Furtwangen orrery, we have the Gotha *Weltmaschine* (Oechslin 8.3) from 1780, and finally the Nuremberg *Weltmaschine* (Oechslin 8.2) constructed along the 1770s and 1780s. These orreries are all different, although the Furtwangen one is probably closest to the one in Gotha. I will make more detailed comparisons during my analysis of the gears.

[1] For biographical details on Hahn, I refer the reader to the chapter on the Ludwigsburg *Weltmaschine* (Oechslin 8.1). The present machine was described by Kistner in the catalogue of the Furtwangen museum published in 1925 [1, p. 55-66]. It is also mentioned by Zinner [4, p. 353] and especially in the 1989 Hahn exhibition catalogue [3, p. 455-457].

[2] Some accounts claim that the orrery was used in the *Collegium illustre* which was in Tübingen, but this seems incorrect.

9.2 The going work

The going work provides a rotation of one turn in 24 hours on arbor 1:

$$\mathrm{V}_1^0 = -1 \tag{9.1}$$

This motion is clockwise as seen from the outside.

It is used to produce the motion of the central arbor 5 of the orrery:

$$\mathrm{V}_2^0 = \mathrm{V}_1^0 \times \left(-\frac{35}{35}\right) \tag{9.2}$$

$$\mathrm{V}_3^0 = \mathrm{V}_2^0 \times \frac{76}{79} \text{ (because of how the wheels mesh)} \tag{9.3}$$

$$\mathrm{V}_4^0 = \mathrm{V}_3^0 \times \frac{57}{58} \text{ (same)} \tag{9.4}$$

$$\mathrm{V}_5^0 = \mathrm{V}_4^0 \times \frac{35}{33} \tag{9.5}$$

and therefore

$$\mathrm{V}_5^0 = \frac{25270}{25201} \tag{9.6}$$

This is the velocity of a sidereal day.[3] The motion of the central arbor 5 is counterclockwise as seen from above. This ratio is the usual ratio used by Hahn for the sidereal day.

9.3 The orrery

The main part of the orrery is a fixed frame (0) containing the gears for the Moon and for the planets up to Mars. The motion Mars, as well as the base sidereal motion, are then used to derive the motions of Jupiter and Saturn.

9.3.1 The first planets and the Moon

The motions of the planets are obtained in a rather simple way, and the orrery is in that matter simpler than that of the Ludwigsburg machine (Oechslin 1.1) from 1769. In the latter, the motions of the planets were obtained by intermittent corrections, and the planets were moving in (simulated) tilted planes. In the Furtwangen orrery, all the planets move in the same horizontal plane and the motions are regular. Venus and the Earth have a simple circular motion. Mercury and Mars, on the other hand, move on eccentric circles that approximate ellipses. Finally, the Moon is revolving around the Earth.

[3] According to Oechslin, the wheel which should have had 79 teeth above actually only has 78 teeth [2, p. 204].

9.3.1.1 The mean motions

We can easily compute these motions. The only input to the fixed frame is the vertical arbor 5 which makes one turn counterclockwise (seen from above) in a sidereal day.

This motion is first used to derive the mean motion of Mercury on tube 9:

$$V_9^0 = V_5^0 \times \left(-\frac{52}{52}\right) \times \left(-\frac{6}{53}\right) \times \left(-\frac{32}{77}\right) \times \left(-\frac{20}{83}\right) \quad (9.7)$$

$$= V_5^0 \times \frac{3840}{338723} = \frac{25270}{25201} \times \frac{3840}{338723} \quad (9.8)$$

$$= \frac{13862400}{1219451189} \quad (9.9)$$

$$P_9^0 = \frac{1219451189}{13862400} = 87.9682\ldots \text{ days} \quad (9.10)$$

This motion is counterclockwise. The same value is given by Oechslin.

The same ratio is used in the Stuttgart globe clock (Oechslin 8.5), and in the orreries of Gotha (Oechslin 8.3) and Nuremberg (Oechslin 8.2).

The motion of Mercury is used to obtain the motion of Venus on tube 13:

$$V_{13}^0 = V_9^0 \times \left(-\frac{54}{50}\right) \times \left(-\frac{29}{80}\right) = V_9^0 \times \frac{783}{2000} \quad (9.11)$$

$$= \frac{13862400}{1219451189} \times \frac{783}{2000} = \frac{935712}{210250205} \quad (9.12)$$

$$P_{13}^0 = \frac{210250205}{935712} = 224.6954\ldots \text{ days} \quad (9.13)$$

The same value is given by Oechslin.

The same ratio is also used in the Stuttgart globe clock (Oechslin 8.5), and in the orreries of Gotha (Oechslin 8.3) and Nuremberg (Oechslin 8.2).

The motion of Venus is used to obtain the motion of the Earth on tube 15:

$$V_{15}^0 = V_{13}^0 \times \left(-\frac{79}{65}\right) \times \left(-\frac{41}{81}\right) = V_{13}^0 \times \frac{3239}{5265} \quad (9.14)$$

$$= \frac{935712}{210250205} \times \frac{3239}{5265} = \frac{473632}{172990675} \quad (9.15)$$

$$P_{15}^0 = \frac{172990675}{473632} = 365.2427\ldots \text{ days} \quad (9.16)$$

The same value is given by Oechslin.

The same ratio is also used in the Stuttgart globe clock (Oechslin 8.5), and in the orreries of Gotha (Oechslin 8.3) and Nuremberg (Oechslin 8.2).

The motion of the Earth is used to obtain the motion of Mars on tube 17:

$$V_{17}^0 = V_{15}^0 \times \left(-\frac{116}{33}\right) \times \left(-\frac{18}{119}\right) = V_{15}^0 \times \frac{696}{1309} \quad (9.17)$$

$$= \frac{473632}{172990675} \times \frac{696}{1309} = \frac{329647872}{226444793575} \quad (9.18)$$

$$P_{17}^0 = \frac{226444793575}{329647872} = 686.9293\ldots \text{ days} \quad (9.19)$$

The same value is given by Oechslin.

The same ratio is used in the Stuttgart globe clock (Oechslin 8.5), the Aschaffenburg globe clock (Oechslin 8.4), and in the Gotha orrery (Oechslin 8.3), but not in the Nuremberg orrery (Oechslin 8.2).

9.3.1.2 The motion of the Moon

The motion of Mercury is also used to obtain the mean motion of the Moon on tube 11:

$$\mathrm{V}_{11}^{0} = \mathrm{V}_{9}^{0} \times \left(-\frac{84}{85}\right) \times \left(-\frac{101}{31}\right) = \mathrm{V}_{9}^{0} \times \frac{8484}{2635} \quad (9.20)$$

$$= \frac{13862400}{1219451189} \times \frac{8484}{2635} = \frac{23521720320}{642650776603} \quad (9.21)$$

$$\mathrm{P}_{11}^{0} = \frac{642650776603}{23521720320} = 27.3215\ldots \text{ days} \quad (9.22)$$

This is an approximation of the tropical month. The same value is given by Oechslin.

The same ratio is used in the orreries of Gotha (Oechslin 8.3) and Nuremberg (Oechslin 8.2), but not in the Stuttgart globe clock (Oechslin 8.5).

The Moon is carried by tube 34 which replicates the motion of tube 11.

9.3.1.3 The anomalies of Mercury and Mars

Mercury and Mars are each excentered on wheels that replicate the motion of a central wheel or tube which is fixed. In the case of Mercury, the fixed wheel is a 20-teeth wheel on the central arbor carrying the Sun, and this arbor is fixed to the cage (frame) containing the various gears that have been described. In the case of Mars, the fixed wheel is a 40-teeth wheel fixed on top of the above-mentioned frame.

Consequently, Mercury and Mars merely oscillate around their mean position with the orbital period of the planet. The lines of apsides of both planets are considered fixed.

9.3.2 Jupiter and its satellites

Jupiter and its satellites is carried on an arm which revolves around the Sun. This arm moves with the period of Jupiter's orbital period. In addition, two horizontal arbors are mounted on this arm, to care for the eccentric motion of Jupiter and for the motion of its satellites. We must therefore consider *three* inputs for the motion of the Jupiter system.

The support arm of Jupiter is carried by tube 19. The motion of this tube is derived from the mean motion of Mars. This tube has the velocity

$$\mathrm{V}_{19}^{0} = \mathrm{V}_{17}^{0} \times \left(-\frac{119}{31}\right) \times \left(-\frac{5}{121}\right) = \mathrm{V}_{17}^{0} \times \frac{595}{3751} \tag{9.23}$$

$$= \frac{329647872}{226444793575} \times \frac{595}{3751} = \frac{329647872}{1427553648235} \tag{9.24}$$

$$\mathrm{P}_{19}^{0} = \frac{1427553648235}{329647872} = 4330.5410\ldots \text{ days} \tag{9.25}$$

The same value is given by Oechslin.

The same ratio is used in the Gotha orrery (Oechslin 8.3) and in the Stuttgart globe clock (Oechslin 8.5).

As mentioned above, two arbors are mounted on the arm supporting Jupiter and its satellites, one (28) for the eccentric motion of Jupiter, and the other (38) for the motion of the satellites of Jupiter. A similar construction was used in the Gotha *Weltmaschine* in 1780, except that the horizontal arbors are replaced by gear trains.

9.3.2.1 The eccentric motion of Jupiter

The eccentric motion of Jupiter is obtained very simply. The system of Jupiter is actually fixed on the frame 29 which rotates excentrically on the support arm 19. The orientation of frame 29 is actually replicating that of a fixed wheel on the fixed central tube of the orrery. Consequently, Jupiter moves on an eccentric circle, an approximation of its elliptic orbit.

9.3.2.2 Jupiter's satellites

The motion of the satellites of Jupiter is obtained from the motion of the arbor 38 mounted on the arm supporting the Jupiter system. The motion of this arbor is derived from the motion of the tube 36, which is itself derived from the fundamental sidereal day motion. We can easily compute the motion of tube 36:

$$\mathrm{V}_{36}^{0} = \mathrm{V}_{5}^{0} \times \left(-\frac{52}{52}\right) \times \left(-\frac{24}{24}\right) \times \left(-\frac{52}{52}\right) = -\mathrm{V}_{5}^{0} = -\frac{25270}{25201} \tag{9.26}$$

In other words, tube 36 makes one turn *clockwise* in one sidereal day.

The motion of tube 36 is then replicated in the motion of arbor 39, which goes through tube 29. And since the orientation of tube 29 is fixed in space, we can merely consider the motion of the four satellites of Jupiter with an input which is that of the sidereal day motion. We have

$$V^0_{42} = V^0_{39} \times \left(-\frac{26}{26}\right) \times \left(-\frac{35}{57}\right) \times \left(-\frac{56}{61}\right) \tag{9.27}$$

$$= V^0_{39} \times \left(-\frac{1960}{3477}\right) = V^0_{36} \times \left(-\frac{1960}{3477}\right) \tag{9.28}$$

$$= \left(-\frac{25270}{25201}\right) \times \left(-\frac{1960}{3477}\right) = \frac{2606800}{4611783} \tag{9.29}$$

$$P^0_{42} = \frac{4611783}{2606800} = 1.7691\ldots \text{ days (Io)} \tag{9.30}$$

The same ratio is used in the *Weltmaschinen* of Gotha (Oechslin 8.3) and Nuremberg (Oechslin 8.2).

It should also be noted that Oechslin mistakenly used the ratio $\frac{36}{57}$ instead of $\frac{35}{57}$ in his computations, and consequently obtained wrong periods for all four satellites of Jupiter. The same mistake was made in the analysis of the Gotha machine, but not in the Nuremberg machine.

Then

$$V^0_{44} = V^0_{42} \times \left(-\frac{67}{68}\right) \times \left(-\frac{45}{89}\right) = V^0_{42} \times \frac{3015}{6052} \tag{9.31}$$

$$= \frac{2606800}{4611783} \times \frac{3015}{6052} = \frac{654958500}{2325875893} \tag{9.32}$$

$$P^0_{44} = \frac{2325875893}{654958500} = 3.5511\ldots \text{ days (Europa)} \tag{9.33}$$

The same ratio is used in the *Weltmaschine* of Gotha (Oechslin 8.3) but not in that of Nuremberg (Oechslin 8.2).

$$V^0_{46} = V^0_{44} \times \left(-\frac{94}{41}\right) \times \left(-\frac{21}{97}\right) = V^0_{44} \times \frac{1974}{3977} \tag{9.34}$$

$$= \frac{654958500}{2325875893} \times \frac{1974}{3977} = \frac{1292888079000}{9250008426461} \tag{9.35}$$

$$P^0_{46} = \frac{9250008426461}{1292888079000} = 7.1545\ldots \text{ days (Ganymede)} \tag{9.36}$$

The same ratio is used in the *Weltmaschine* of Gotha (Oechslin 8.3) but not in that of Nuremberg (Oechslin 8.2).

$$V^0_{48} = V^0_{46} \times \left(-\frac{66}{53}\right) \times \left(-\frac{21}{61}\right) = V^0_{46} \times \frac{1386}{3233} \tag{9.37}$$

$$= \frac{1292888079000}{9250008426461} \times \frac{1386}{3233} = \frac{162903897954000}{2718661567522583} \tag{9.38}$$

$$P^0_{48} = \frac{2718661567522583}{162903897954000} = 16.6887\ldots \text{ days (Calliso)} \tag{9.39}$$

The same ratio is used in the *Weltmaschine* of Gotha (Oechslin 8.3) but not in that of Nuremberg (Oechslin 8.2).

The periods of Europa, Ganymede and Callisto in the Nuremberg *Weltmaschine* are somewhat less accurate than those of the Furtwangen orrery, even when the Nuremberg machine is corrected of its flaws.

9.3.3 Saturn and its satellites

The structure of the system of Saturn is similar to that of Jupiter. There is a similar arm, and also two similar horizontal arbors used to provide the eccentric motion of Saturn and the motion of its satellites.

The support arm of Saturn is carried by tube 22. Its motion is derived from tube 19, the support arm of Jupiter. When tube 19 rotates, it carries a wheel on arbor 21 which meshes with a wheel on a fixed central tube. Arbor 21 then rotates and this rotation is transmitted to tube 22. We therefore first consider the motion of tube 22 in the reference frame of tube 19:

$$\mathrm{V}_{22}^{19} = \mathrm{V}_{20}^{19} \times \left(-\frac{103}{68}\right) \times \left(-\frac{41}{104}\right) = \mathrm{V}_{20}^{19} \times \frac{4223}{7072} \tag{9.40}$$

where tube 20 is the fixed central tube.

$$= -\mathrm{V}_{19}^{0} \times \frac{4223}{7072} = -\frac{329647872}{1427553648235} \times \frac{4223}{7072} \tag{9.41}$$

$$= -\frac{43503217608}{315489356259935} \tag{9.42}$$

$$\mathrm{V}_{22}^{0} = \mathrm{V}_{22}^{19} + \mathrm{V}_{19}^{0} = \mathrm{V}_{19}^{0} \times \left(1 - \frac{4223}{7072}\right) = \mathrm{V}_{19}^{0} \times \frac{2849}{7072} \tag{9.43}$$

$$= \frac{329647872}{1427553648235} \times \frac{2849}{7072} = \frac{2668087464}{28680850569085} \tag{9.44}$$

$$\mathrm{P}_{22}^{0} = \frac{28680850569085}{2668087464} = 10749.5915\ldots \text{ days} \tag{9.45}$$

The same value is obtained by Oechslin.

The same ratio is used in the Gotha orrery (Oechslin 8.3), but not in other machines described by Oechslin.

As mentioned above, two arbors are mounted on the arm supporting Saturn and its satellites, one (31) for the eccentric motion of Saturn, and the other (51) for the motion of the satellites of Saturn. A similar construction was used in the Gotha *Weltmaschine* in 1780, except that the horizontal arbors are replaced by gear trains.

9.3.3.1 The eccentric motion of Saturn

Like for Jupiter, the system of Saturn is fixed on the frame 32 which rotates excentrically on the support arm 22. The orientation of frame 32 is also replicating that of a fixed wheel on the fixed central tube of the orrery. Consequently, Saturn moves on an eccentric circle, an approximation of its elliptic orbit.

9.3.3.2 Saturn's satellites

The motion of the satellites of Saturn is obtained from the motion of the arbor 51 mounted on the arm supporting the Saturn system. The motion of this arbor is derived from the motion of the tube 49, which has the same motion as the tube 36 used for the satellites of Jupiter.

$$\mathrm{V}^0_{49} = \mathrm{V}^0_{36} = -\frac{25270}{25201} \tag{9.46}$$

In other words, tube 49 also makes one turn *clockwise* in one sidereal day. The motion of tube 49 is then replicated in the motion of arbor 52, which goes through tube 29. And since the orientation of tube 32 is fixed in space, we can merely consider the motion of the five satellites of Saturn with an input which is that of the sidereal day motion. If we take the actual teeth numbers as given on Oechslin's drawing, we find that the motions of the five satellites are the following ones:

For Tethys:

$$\mathrm{V}^0_{55} = \mathrm{V}^0_{52} \times \left(-\frac{20}{78}\right) \times \left(-\frac{86}{33}\right) = \mathrm{V}^0_{52} \times \frac{860}{1287} \tag{9.47}$$

$$= \frac{25270}{25201} \times \frac{860}{1287} = \frac{21732200}{32433687} \tag{9.48}$$

$$\mathrm{P}^0_{55} = \frac{32433687}{21732200} = 1.4924\ldots \text{ days} \tag{9.49}$$

For Dione:

$$\mathrm{V}^0_{57} = \mathrm{V}^0_{55} \times \left(-\frac{22}{46}\right) \times \left(-\frac{49}{34}\right) = \mathrm{V}^0_{55} \times \frac{539}{782} \tag{9.50}$$

$$= \frac{21732200}{32433687} \times \frac{539}{782} = \frac{532438900}{1152870147} \tag{9.51}$$

$$\mathrm{P}^0_{57} = \frac{1152870147}{532438900} = 2.1652\ldots \text{ days} \tag{9.52}$$

For Rhea:

$$\mathrm{V}^0_{59} = \mathrm{V}^0_{57} \times \left(-\frac{27}{32}\right) \times \left(-\frac{23}{32}\right) = \mathrm{V}^0_{57} \times \frac{621}{1024} \tag{9.53}$$

$$= \frac{532438900}{1152870147} \times \frac{621}{1024} = \frac{399329175}{1425771776} \tag{9.54}$$

$$\mathrm{P}^0_{59} = \frac{1425771776}{399329175} = 3.5704\ldots \text{ days} \tag{9.55}$$

For Titan:

$$\mathrm{V}^0_{61} = \mathrm{V}^0_{59} \times \left(-\frac{40}{36}\right) \times \left(-\frac{12}{47}\right) = \mathrm{V}^0_{59} \times \frac{40}{141} \tag{9.56}$$

$$= \frac{399329175}{1425771776} \times \frac{40}{141} = \frac{665548625}{8376409184} \tag{9.57}$$

$$\mathrm{P}^0_{61} = \frac{8376409184}{665548625} = 12.5857\ldots \text{ days} \tag{9.58}$$

For Iapetus:

$$\mathrm{V}^0_{63} = \mathrm{V}^0_{61} \times \left(-\frac{68}{39}\right) \times \left(-\frac{8}{69}\right) = \mathrm{V}^0_{61} \times \frac{544}{2691} \tag{9.59}$$

$$= \frac{665548625}{8376409184} \times \frac{544}{2691} = \frac{665548625}{41435509401} \tag{9.60}$$

$$\mathrm{P}^0_{63} = \frac{41435509401}{665548625} = 62.2576\ldots \text{ days} \tag{9.61}$$

These values are all wrong by a factor of about 1.27. The actual periods are 1.9 days (Tethys), 2.7 days (Dione), 4.5 days (Rhea), 16 days (Titan) and 79 days (Iapetus).

It does however appear that there are two possible causes to these discrepancies. First, Oechslin's plan gives a ratio of 86/33 for Tethys, but in his computations he uses the ratio 68/33 (values marked "(a)" below), and therefore it is possible that the 86-teeth wheel should have had 68 teeth.[4] This problem is the same in the Gotha *Weltmaschine* from 1780 (Oechslin 8.3) where the system of the satellites of Saturn uses exactly the same ratios. The wrong ratio used in the motion of Tethys propagates to the other satellites, because the motion of each satellite is obtained from the previous one.

It seems that this problem in the Furtwangen planetarium was not noticed by Hahn and made its way to the Gotha *Weltmaschine*.

There is however another possibility. In the Furtwangen orrery, one of the ratios used in the motion of Tethys is $\frac{20}{78}$ and if we replace it by $\frac{20}{98}$, we also obtain much more accurate periods (values marked "(b)" below). This possibility also applies to the Gotha machine, the only difference being that the ratio $\frac{20}{78}$ was replaced by $\frac{10}{39}$.

So, the periods given by Oechslin are those obtained when Tethys uses the ratio 68/33. The motions of the five satellites are then the following, where I have added "(a)" to distinguish this choice from the second one "(b)" below.

[4] See [2, p. 204] on this error.

For Tethys:

$$\mathrm{V}^0_{55}(a) = \mathrm{V}^0_{52} \times \left(-\frac{20}{78}\right) \times \left(-\frac{68}{33}\right) = \mathrm{V}^0_{52} \times \frac{680}{1287} \tag{9.62}$$

$$= \frac{25270}{25201} \times \frac{680}{1287} = \frac{17183600}{32433687} \tag{9.63}$$

$$\mathrm{P}^0_{55}(a) = \frac{32433687}{17183600} = 1.8874\ldots \text{ days} \tag{9.64}$$

For Dione:

$$\mathrm{V}^0_{57}(a) = \mathrm{V}^0_{55}(a) \times \left(-\frac{22}{46}\right) \times \left(-\frac{49}{34}\right) = \mathrm{V}^0_{55}(a) \times \frac{539}{782} \tag{9.65}$$

$$= \frac{17183600}{32433687} \times \frac{539}{782} = \frac{24764600}{67815891} \tag{9.66}$$

$$\mathrm{P}^0_{57}(a) = \frac{67815891}{24764600} = 2.7384\ldots \text{ days} \tag{9.67}$$

For Rhea:

$$\mathrm{V}^0_{59}(a) = \mathrm{V}^0_{57}(a) \times \left(-\frac{27}{32}\right) \times \left(-\frac{23}{32}\right) = \mathrm{V}^0_{57}(a) \times \frac{621}{1024} \tag{9.68}$$

$$= \frac{24764600}{67815891} \times \frac{621}{1024} = \frac{9286725}{41934464} \tag{9.69}$$

$$\mathrm{P}^0_{59}(a) = \frac{41934464}{9286725} = 4.5155\ldots \text{ days} \tag{9.70}$$

For Titan:

$$\mathrm{V}^0_{61}(a) = \mathrm{V}^0_{59}(a) \times \left(-\frac{40}{36}\right) \times \left(-\frac{12}{47}\right) = \mathrm{V}^0_{59}(a) \times \frac{40}{141} \tag{9.71}$$

$$= \frac{9286725}{41934464} \times \frac{40}{141} = \frac{15477875}{246364976} \tag{9.72}$$

$$\mathrm{P}^0_{61}(a) = \frac{246364976}{15477875} = 15.9172\ldots \text{ days} \tag{9.73}$$

For Iapetus:

$$\mathrm{V}^0_{63}(a) = \mathrm{V}^0_{61}(a) \times \left(-\frac{68}{39}\right) \times \left(-\frac{8}{69}\right) = \mathrm{V}^0_{61}(a) \times \frac{544}{2691} \tag{9.74}$$

$$= \frac{15477875}{246364976} \times \frac{544}{2691} = \frac{526247750}{41435509401} \tag{9.75}$$

$$\mathrm{P}^0_{63}(a) = \frac{41435509401}{526247750} = 78.7376\ldots \text{ days} \tag{9.76}$$

If instead we replace the ratio 20/78 by 20/98, we obtain the following values, marked "(b)" in order to distinguish them from the previous ones:

For Tethys:

$$V^0_{55}(b) = V^0_{52} \times \left(-\frac{20}{98}\right) \times \left(-\frac{86}{33}\right) = V^0_{52} \times \frac{860}{1617} \tag{9.77}$$

$$= \frac{25270}{25201} \times \frac{860}{1617} = \frac{3104600}{5821431} \tag{9.78}$$

$$P^0_{55}(b) = \frac{5821431}{3104600} = 1.8750\ldots \text{ days} \tag{9.79}$$

For Dione:

$$V^0_{57}(b) = V^0_{55}(b) \times \left(-\frac{22}{46}\right) \times \left(-\frac{49}{34}\right) = V^0_{55}(b) \times \frac{539}{782} \tag{9.80}$$

$$= \frac{3104600}{5821431} \times \frac{539}{782} = \frac{10866100}{29560773} \tag{9.81}$$

$$P^0_{57}(b) = \frac{29560773}{10866100} = 2.7204\ldots \text{ days} \tag{9.82}$$

For Rhea:

$$V^0_{59}(b) = V^0_{57}(b) \times \left(-\frac{27}{32}\right) \times \left(-\frac{23}{32}\right) = V^0_{57}(b) \times \frac{621}{1024} \tag{9.83}$$

$$= \frac{10866100}{29560773} \times \frac{621}{1024} = \frac{24448725}{109674752} \tag{9.84}$$

$$P^0_{59}(b) = \frac{109674752}{24448725} = 4.4859\ldots \text{ days} \tag{9.85}$$

For Titan:

$$V^0_{61}(b) = V^0_{59}(b) \times \left(-\frac{40}{36}\right) \times \left(-\frac{12}{47}\right) = V^0_{59}(b) \times \frac{40}{141} \tag{9.86}$$

$$= \frac{24448725}{109674752} \times \frac{40}{141} = \frac{40747875}{644339168} \tag{9.87}$$

$$P^0_{61}(b) = \frac{644339168}{40747875} = 15.8128\ldots \text{ days} \tag{9.88}$$

For Iapetus:

$$V^0_{63}(b) = V^0_{61}(b) \times \left(-\frac{68}{39}\right) \times \left(-\frac{8}{69}\right) = V^0_{61}(b) \times \frac{544}{2691} \tag{9.89}$$

$$= \frac{40747875}{644339168} \times \frac{544}{2691} = \frac{13582625}{1062448959} \tag{9.90}$$

$$P^0_{63}(b) = \frac{1062448959}{13582625} = 78.2211\ldots \text{ days} \tag{9.91}$$

It isn't clear which of these two solutions is the right one.

9.4 References

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